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Coordinate systems

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NAME

Numerical Atmospheric-Dispersion Modelling Environment

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1 Introduction

A number of different horizontal and vertical coordinate systems are used. These are used both internally within the model and for input and output. Code supporting use of these coordinate systems is provided in the coordinate system module.

In defining the coordinate systems and in computing the transformations between the systems it is assumed that the Earth is essentially spherical[1] with radius R_E. More precisely, we take the mean sea level surface and the reference geoid (a surface of constant gravitational and centrifugal potential) to be coincident with the surface of a sphere of radius R_E. R_E is taken to be 6,371,229 m (Meteorological Office (1991, page 99); Wilson (2000)). We also assume that the region of interest is thin relative to R_E.

2 Horizontal coordinate systems

Six different types of coordinate system are supported:

- Coordinate systems based on a latitude-longitude coordinate system with arbitrary orientation, i.e. arbitrary location for the latitude-longitude coordinate system's north pole and for the 'third' Euler angle representing a rotation about the latitude-longitude coordinate system's north pole (we use the phrase 'coordinate systems based on a latitude-longitude coordinate system' rather than 'latitude-longitude coordinate systems' because of the possibility of a non-zero origin offset and non-standard choice of units)
- Cartesian coordinate systems using a stereographic projection
- Polar coordinate systems using a stereographic projection
- Cartesian coordinate systems using a transverse Mercator projection
- Polar coordinate systems using a transverse Mercator projection
- Cartesian coordinate systems using a conformal Lambert projection.

We use the following symbols for the coordinates:

- For latitude-longitude based coordinate systems the first and second coordinates are written as λ and ϕ, and are related to longitude and latitude respectively.
- For Cartesian coordinate systems the first and second coordinates are written as x and y.
- For polar coordinate systems the first and second coordinates are written as r and θ, and are related to distance from the origin and angle about the origin respectively.

We will also use (x, y) from time to time to denote a generic horizontal coordinate system.

All horizontal coordinates are defined to be constant on vertical lines normal to the sphere of radius R_E. This is trivially true for latitude-longitude systems and is true for stereographic and transverse mercator projections because the projections are defined for points on the sphere and then extended consistently to points at different radii.

For each coordinate system there is the option of specifying an origin offset (i.e. an offset for the position of the point (0,0) for latitude-longitude based coordinate systems and Cartesian coordinate systems, and the position of the point $r = 0$ for polar coordinate systems), the units used, and for polar coordinate systems, the direction of the line $\theta = 0$. For transverse Mercator projection based coordinate systems one can also specify a scale factor $s > 0$ (defined below). Units can be negative so that one can change the direction in which a coordinate increases. When units are positive we adopt the following conventions:

- λ and ϕ increase to the east and north.
- The positive y axis is rotated 90 degrees anticlockwise from the positive x axis.
- θ increases anticlockwise.

[1] In future we could support some coordinate systems defined using other Earth shapes e.g. for accurate interpretation of national grid coordinates. If we do this, then we could retain the spherical Earth assumption internally within NAME III and simply regard these coordinates as being rather complicated functions of latitude and longitude.

There is some redundancy in the variables we have used to define the various coordinate systems. For example, for latitude-longitude based coordinate systems, the inclusion of both e_3 and λ_o (defined below) is redundant. We have retained this redundancy in order to achieve a uniformity of approach between the different types of coordinate systems and because it is convenient in defining specific coordinate systems.

2.1 Latitude-longitude based coordinate systems

To define a latitude-longitude based coordinate system we need to specify the longitude and latitude of the coordinate system's north pole in the standard latitude-longitude coordinate system and the angle of rotation of the coordinate system about its north pole. We will denote these three quantities by λ_p, ϕ_p and e_3, the notation e_3 reflecting the fact that it can be thought of as the third Euler angle of the transformation from the standard latitude-longitude coordinate system. e_3 is defined so that it is zero when the coordinate system's zero longitude line passes through the true south pole. A positive value of e_3 means that, standing at the coordinate system's north pole and looking down, the coordinate system is rotated anticlockwise relative to its orientation for a zero value. It is convenient at times to work with colatitudes instead of latitudes — the colatitude is the angle from the north pole and will be denoted by $\theta = \pi/2 - \phi$ (so that, for example, $\theta_p = \pi/2 - \phi_p$).

The three Euler rotations (see e.g. Arnold (1978, page 149)) needed to get from the standard latitude-longitude coordinate system to the rotated coordinate system are defined as follows (note we are considering the Earth to be fixed and are rotating the coordinate system, not vice versa). The first rotation rotates the coordinate system by λ_p about the true north pole (positive λ_p indicating an anticlockwise rotation looking down on the north pole). The second rotation takes the coordinate system's north pole away from the true north pole down the zero longitude line (i.e. the position of the zero longitude line after the first rotation) by an angle θ_p. The third rotation rotates the coordinate system by e_3 about its north pole (positive e_3 indicating an anticlockwise rotation looking down on the coordinate system's north pole).

If $\theta_p = 0$ or π then the definitions given in the first paragraph of this section are ill-defined. However we can provide a natural interpretation using continuity or using the Euler angle approach in the second paragraph of this section. If $\theta_p = 0$, the coordinate system is the standard latitude-longitude coordinate system rotated by an angle $\lambda_p + e_3$ (anticlockwise looking down on the north pole), i.e. the coordinate system's north pole coincides with the true north pole and the system's zero longitude line is at a true longitude of $\lambda_p + e_3$. If $\theta_p = \pi$, the coordinate system's north pole is at the true south pole and the system's zero longitude line is at a true longitude of $\pi + \lambda_p - e_3$.

We can now define a coordinate system with arbitrary origin offset and units. For origin offset (λ_o, ϕ_o) and units (λ_u, ϕ_u) we define the coordinate values (λ, ϕ) in terms of the longitude and latitude $(\hat{\lambda}, \hat{\phi})$ in the rotated latitude-longitude coordinate system by

$$\lambda = (\hat{\lambda} - \lambda_o)/\lambda_u \quad \text{and} \quad \phi = (\hat{\phi} - \phi_o)/\phi_u.$$

Here all quantities are in radians except the coordinate values (λ, ϕ) which of course depend on the choice of units (λ_u, ϕ_u). $\lambda_u \lambda$ and $\hat{\phi}$ should lie in $[-\pi, \pi]$ and $[-\pi/2, \pi/2]$ respectively.

2.2 Stereographic projection based coordinate systems

To define a stereographic projection (see e.g. Frankel (2004, page 21); Porteous (1981, page 170); Weisstein (2003, page 2857-2858)) we consider a rotated latitude-longitude coordinate system defined by λ_p, ϕ_p and e_3. We then consider the tangent plane at the north pole of the rotated latitude-longitude coordinate system and project onto this plane from the south pole of the rotated latitude-longitude coordinate system. We choose a Cartesian coordinate system in the tangent plane with the positive x and y axes being the projection of the $\pi/2$ and π longitude lines in the rotated latitude-longitude coordinate system.

Stereographic projections are sometimes defined as a projection onto the equatorial plane (this definition is used by Porteous) but we use the tangent plane at the north pole so that the projection naturally preserves distances at that point.

We can now define a Cartesian coordinate system in the tangent plane with arbitrary origin offset and units. For origin offset (x_o, y_o) and units (x_u, y_u) we define the coordinate values (x, y) in terms of the coordinates $(\hat{x}, \hat{y})$ in the basic Cartesian coordinate system defined above (i.e. the system with origin at the tangent point) by

$$x = \frac{\hat{x} - x_o}{x_u} \quad \text{and} \quad y = \frac{\hat{y} - y_o}{y_u}.$$

Here all quantities are in metres except the coordinate values (x, y) which of course depend on the choice of units (x_u, y_u).

We can also define a polar coordinate system in the tangent plane with arbitrary origin offset, θ origin and units. For origin offset (x_o, y_o), θ origin θ_o and units (r_u, θ_u) we define the coordinate values (r, θ) in terms of the coordinates $(\hat{x}, \hat{y})$ in the basic Cartesian coordinate system defined above (i.e. the system with origin at the tangent point) by

$$r = \frac{\sqrt{(\hat{x} - x_o)^2 + (\hat{y} - y_o)^2}}{r_u}$$

and

$$\theta = \frac{\arctan_2(\hat{y} - y_o, \hat{x} - x_o) - \theta_o}{\theta_u}.$$

where $\arctan_2$ is the function defined by $\arctan_2(r \sin(\alpha), r \cos(\alpha)) = \alpha$ for $\alpha \in (-\pi, \pi]$. Here all quantities are in metres or radians except the coordinate values (r, θ) which of course depend on the choice of units (r_u, θ_u). $\theta_u \theta$ should lie in $[-\pi, \pi]$.

2.3 Transverse Mercator projection based coordinate systems

To define the standard Mercator projection (see e.g. Weisstein (2003, page 1894-1895)) we consider a cylinder aligned with the Earth's axis and touching the equator. Each longitude line is projected onto the cylinder from the centre of the Earth, but the image of each longitude line is then stretched/compressed (compressed actually) in the direction parallel to the Earth's axis so that the transformation is conformal (angle preserving). We choose the y axis to be the line parallel to the axis of the cylinder and passing through the point with zero latitude and longitude, and the x axis to be the line of contact between the Earth and the cylinder. x and y increase in going from the point with zero latitude and longitude to the east and north respectively.

A transverse Mercator projection is the same projection but with the cylinder aligned at right angles to the Earth's axis. We can define a transverse Mercator projection by defining the longitude λ_{to} and latitude ϕ_{to} of the 'true origin' of the projection in the standard latitude-longitude coordinate system. We then take the axis of the cylinder to go through the equator at longitude $\lambda_{to} + \pi/2$. We choose the x axis to be the line parallel to the axis of the cylinder and passing through the true origin and the y axis to be the line of contact between the Earth and the cylinder. x and y increase in going from the true origin to the east and north respectively.

As in §2.1, the above is ill-defined if $\phi_{to} = \pm\pi/2$ but there is a natural interpretation by continuity or by rewording the definition to be 'the standard Mercator projection applied in the latitude-longitude coordinate system with $\lambda_p = \lambda_{to} + \pi/2$, $\phi_p = 0$ and $e_3 = -\phi_{to}$ but with the x and y axes rotated $\pi/2$ anticlockwise'.

The above projection preserves distances on the line of contact between the Earth and cylinder, but is stretched away from that line. Sometimes a scale factor s is introduced to compress the projection and ensure that distances are correct in some average sense over a certain area.

We can now define a Cartesian coordinate system on the cylinder with arbitrary scale factor, origin offset and units. For scale factor s, origin offset (x_o, y_o) and units (x_u, y_u) we define the coordinate values (x, y) in terms of the coordinates $(\hat{x}, \hat{y})$ in the basic Cartesian coordinate system defined above (i.e. the system with origin at the true origin and unit scale factor) by

$$x = \frac{s\hat{x} - x_o}{x_u} \quad \text{and} \quad y = \frac{s\hat{y} - y_o}{y_u}.$$

Here all quantities are in metres except the coordinate values (x, y) which of course depend on the choice of units (x_u, y_u). $y_u y$ should lie in $[-\pi s R_E, \pi s R_E]$. Note that the origin offset (x_o, y_o) is defined in terms of a coordinate system which has been scaled using the scaling factor s (but which has not been scaled using x_u and y_u).

We can also define a polar coordinate system on the cylinder with arbitrary scale factor, origin offset, θ origin and units. For scale factor s, origin offset (x_o, y_o), θ origin θ_o and units (r_u, θ_u) we define the coordinate values (r, θ) in terms of the coordinates $(\hat{x}, \hat{y})$ in the basic Cartesian coordinate system defined above (i.e. the system with origin at the true origin and unit scale factor) by

$$r = \frac{\sqrt{(s\hat{x} - x_o)^2 + (s\hat{y} - y_o)^2}}{r_u}$$

and

$$\theta = \frac{\arctan_2(s\hat{y} - y_o, s\hat{x} - x_o) - \theta_o}{\theta_u}.$$

Here all quantities are in metres or radians except the coordinate values (r, θ) which of course depend on the choice of units (r_u, θ_u). $y_u y$ should lie in $[-\pi s R_E, \pi s R_E]$ and $\theta_u \theta$ should lie in $[-\pi, \pi]$.

2.4 Conformal Lambert projection based coordinate systems

A conformal Lambert projection conceptually places a cone over the Earth and projects the earth's surface on to it (see e.g. Weisstein (2003, page 1022)). It is a conformal projection meaning that angles are preserved. In order to define a particular conformal Lambert projection, two (or sometimes one) standard parallels are given. We will denote these latitudes by ϕ_1 and ϕ_2. These are the latitudes at which the projection is given unit scaling and near these standard parallels is where there is the least distortion. We will denote these latitudes by ϕ_1 and ϕ_2. The other parallels in the projection have a scale factor < 1 between the two standard parallels, and > 1 outside of them. This is how the conformal Lambert projection differs from the secant cone projection which has equally spaced parallels, making it not conformal.

In order to fully define a conformal Lambert projection, a reference or standard longitude is also required, denoted by λ_0. This is the longitude line which is parallel to the y axis in the projection. The origin of the projection may or may not lie along this reference longitude but is usually near it.

A reference latitude is also given. This corresponds to the latitude at which the cone intersects the earth. Together with a scale factor (which would be one to define a conformal Lambert projection with one standard parallel), this could be used to define a conformal Lambert system instead of the two standard parallels, but with the standard parallels known, the reference latitude is not necessary. It is common for the reference latitude to be equal to the latitude of the origin of the projection.

3 Transformations between different horizontal coordinate systems

It is straightforward to account for the scale factor, the origin offsets, the θ origin and the units and to make transformations between Cartesian and polar coordinate systems in the same projection. Hence we do not consider such matters here. For all the coordinate systems considered in this section we assume that the scale factor is unity, that the origin offsets and the θ origin are zero, and that the units are radians and/or metres.

3.1 Transformations between two latitude-longitude coordinate systems

For any latitude-longitude coordinate system defined by λ_p, θ_p and e_3 (in this section it is convenient to use colatitude rather than latitude to define the coordinate systems) we associate a right-handed 3-d Cartesian coordinate system (X, Y, Z) centred at the centre of the Earth with the Z axis passing through the north pole and the X axis passing through $(\lambda, \phi) = (0, 0)$. We measure X, Y and Z in metres. Then we have

$$X = R_E \cos(\phi)\cos(\lambda), \quad Y = R_E \cos(\phi)\sin(\lambda), \quad Z = R_E \sin(\phi)$$

and

$$\phi = \arctan_2\left(Z, \sqrt{X^2 + Y^2}\right), \quad \lambda = \arctan_2(Y, X).$$

The use of $\arctan_2$ has advantages over alternative formulations in terms of $\arcsin$ and/or $\arccos$. The ambiguity caused by the multivalued nature of $\arcsin$ and $\arccos$ is removed and the loss of precision which occurs where the gradients of $\arcsin$ and $\arccos$ are large is avoided.

One can then transform between two latitude-longitude coordinate systems defined by λ_p, θ_p and e_3 and by λ'_p, θ'_p and e'_3 by transforming in steps via the following sequence of coordinate systems:

- latitude-longitude coordinate system defined by λ_p, θ_p, e_3
- latitude-longitude coordinate system defined by λ_p, θ_p, 0
- X, Y, Z coordinate system corresponding to λ_p, θ_p, 0
- X, Y, Z coordinate system corresponding to λ_p, 0, 0
- X, Y, Z coordinate system corresponding to λ'_p, 0, 0

- X, Y, Z coordinate system corresponding to λ'_p, θ'_p, 0
- latitude-longitude coordinate system defined by λ'_p, θ'_p, 0
- latitude-longitude coordinate system defined by λ'_p, θ'_p, e'_3

The steps involved are the addition of e_3 to the longitude, conversion to a 3-d Cartesian coordinate system, three successive rotations of the coordinate system by $-\theta_p$, $\lambda'_p - \lambda_p$ and θ'_p about the Y, Z and Y axes (measured anti-clockwise when looking down the positive axis towards the origin), conversion to a latitude-longitude coordinate system, and subtraction of e'_3 from the longitude. If one system is the standard latitude-longitude coordinate system then we can simplify the procedure. If, for example, the system defined by λ'_p, θ'_p and e'_3 is the standard latitude-longitude coordinate system (i.e. $\lambda'_p = \theta'_p = e'_3 = 0$), then we can use the following sequence of coordinate systems:

- latitude-longitude coordinate system defined by λ_p, θ_p, e_3
- latitude-longitude coordinate system defined by λ_p, θ_p, 0
- X, Y, Z coordinate system corresponding to λ_p, θ_p, 0
- X, Y, Z coordinate system corresponding to λ_p, 0, 0
- latitude-longitude coordinate system defined by λ_p, 0, 0
- latitude-longitude coordinate system defined by 0, 0, 0

The steps involved are the addition of e_3 to the longitude, conversion to a 3-d Cartesian system, a rotation of the coordinate system by $-\theta_p$ about the Y axis (measured anti-clockwise when looking down the positive axis towards the origin), conversion to a latitude-longitude coordinate system, and addition of λ_p to the longitude.

Note that, if the calculated latitude is $\pm\pi/2$, then the longitude calculation gives $\arctan_2(0, 0)$ and is ill-defined. This requires some care in the numerical implementation.

3.2 Transformations between a latitude-longitude coordinate system and a stereographic projection based Cartesian coordinate system

We consider only the case where the values of λ_p, ϕ_p and e_3 are the same for the two coordinate systems. More general cases can be treated by combining the results here with the transformations in §3.1.

Consider a point $\mathbf{P}$ on the sphere and its projection $\mathbf{P}'$ on the tangent plane and consider the coordinates of these points in the 3-d Cartesian coordinate system introduced in §3.1. The coordinates of $\mathbf{P}$ will be written (X, Y, Z). Then the longitude and latitude of $\mathbf{P}$, λ and ϕ, are related to (X, Y, Z) by

$$X = R_E \cos(\phi)\cos(\lambda), \quad Y = R_E \cos(\phi)\sin(\lambda), \quad Z = R_E \sin(\phi)$$

and

$$\phi = \arctan_2\left(Z, \sqrt{X^2 + Y^2}\right), \quad \lambda = \arctan_2(Y, X).$$

The coordinates of $\mathbf{P}'$ will be written (X', Y', R_E). Then the coordinates (x, y) in the stereographic projection based (2-d) Cartesian coordinate system are related to (X', Y') by

$$x = Y', \quad -y = X'.$$

It is clear from figure 1 that (X', Y') can be expressed as either

$$(X', Y') = \frac{2R_E}{R_E + Z}(X, Y)$$

or

$$|(X', Y')| = 2R_E \tan(\theta/2)$$

with (X', Y') parallel to (X, Y). Hence the transfomation from (λ, ϕ) to (x, y) can be expressed as

$$(x, y) = \frac{2R_E \cos(\phi)}{1 + \sin(\phi)}(\sin(\lambda), -\cos(\lambda))$$

or

$$(x, y) = 2R_E \tan(\theta/2)(\sin(\lambda), -\cos(\lambda)).$$

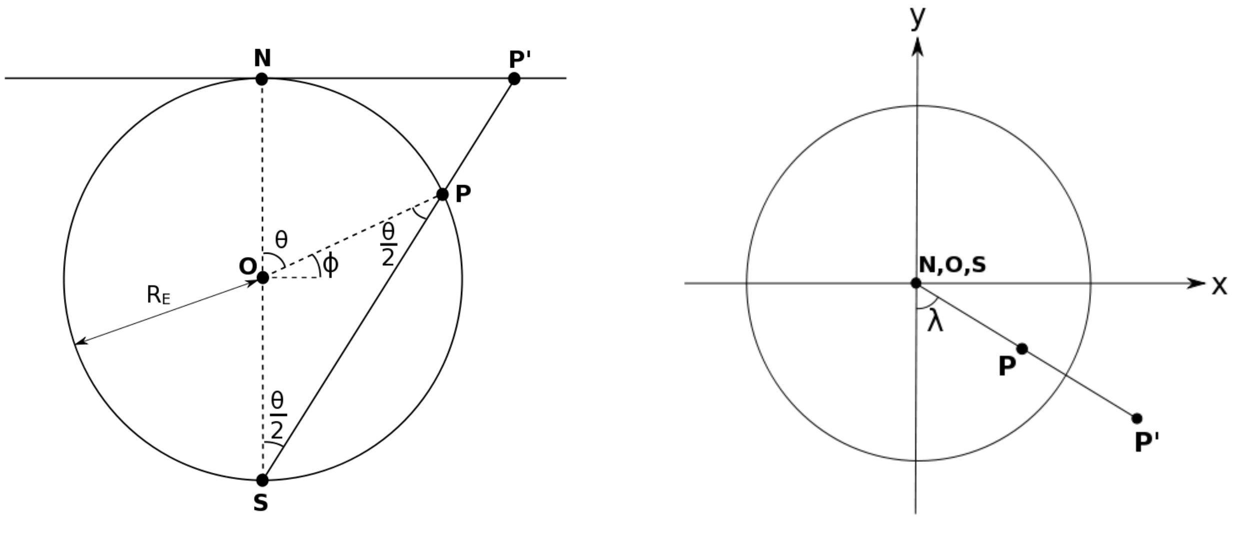


Figure 1: Illustration of a stereographic projection.

The inverse transformation from (x, y) to (λ, ϕ) is most easily derived using the second of these and is given by

$$\phi = \frac{\pi}{2} - 2\arctan\left(\frac{\sqrt{x^2+y^2}}{2R_E}\right), \quad \lambda = \mathrm{arctan}_2(x, -y).$$

Note that, if $\phi = -\pi/2$, then the point is projected to infinity and it is impossible to calculate the corresponding x and y. In addition, if $\phi = \pi/2$ (i.e. $x = y = 0$), then the calculation of λ gives $\mathrm{arctan}_2(0, 0)$ and is ill-defined. This requires some care in the numerical implementation.

In the stereographic projection, infinitesimsal distances are expanded by a factor of $2/(1 + \sin(\phi))$ relative to those on the sphere.

3.3 Transformations between a latitude-longitude coordinate system and a transverse Mercator projection based Cartesian coordinate system

We consider only the case where the latitude-longitude coordinate system is the standard latitude-longitude coordinate system. More general cases can be treated by combining the results here with the transformations in §3.1.

For the standard Mercator projection we have

$$x = R_E\lambda \quad \text{and} \quad y = R_E \log\left(\tan\left(\frac{\phi}{2} + \frac{\pi}{4}\right)\right)$$

with inverse transformation

$$\lambda = \frac{x}{R_E} \quad \text{and} \quad \phi = 2\arctan\left(\exp\left(\frac{y}{R_E}\right)\right) - \frac{\pi}{2}$$

where λ and ϕ are the longitude and latitude in the standard latitude-longitude coordinate system (see e.g. Weisstein (2003, page 1894-1895)). For the transverse Mercator projection we can use this formula in the form

$$-y = R_E\lambda' \quad \text{and} \quad x = R_E \log\left(\tan\left(\frac{\phi'}{2} + \frac{\pi}{4}\right)\right)$$

with inverse transformation

$$\lambda' = \frac{-y}{R_E} \quad \text{and} \quad \phi' = 2\arctan\left(\exp\left(\frac{x}{R_E}\right)\right) - \frac{\pi}{2}$$

where λ' and ϕ' are the longitude and latitude in the latitude-longitude coordinate system with $\lambda_p = \lambda_{to} + \pi/2$, $\phi_p = 0$ and $e_3 = -\phi_{to}$. This is illustrated in figure 2. Using the results on transforming between two latitude-longitude coordinate systems given in §3.1 above, we have

$$\lambda' = \arctan_2(-\sin(\phi), \cos(\phi)\cos(\lambda - \lambda_{to})) + \phi_{to}$$

and

$$\phi' = \arctan_2\left(\cos(\phi)\sin(\lambda - \lambda_{to}), \sqrt{\cos^2(\phi)\cos^2(\lambda - \lambda_{to}) + \sin^2(\phi)}\right)$$

with inverse transformation

$$\lambda = \arctan_2(\sin(\phi'), \cos(\phi')\cos(\lambda' - \phi_{to})) + \lambda_{to}$$

and

$$\phi = \arctan_2\left(-\cos(\phi')\sin(\lambda' - \phi_{to}), \sqrt{\cos^2(\phi')\cos^2(\lambda' - \phi_{to}) + \sin^2(\phi')}\right).$$

The expressions for ϕ' and ϕ can be simplified in various ways, e.g.

$$\begin{aligned}
\phi &= \arctan_2\left(-\cos(\phi')\sin(\lambda' - \phi_{to}), \sqrt{\cos^2(\phi')\cos^2(\lambda' - \phi_{to}) + \sin^2(\phi')}\right) \\
&= \arctan_2\left(-\cos(\phi')\sin(\lambda' - \phi_{to}), \sqrt{1 - \cos^2(\phi')\sin^2(\lambda' - \phi_{to})}\right) \\
&= \arcsin(-\cos(\phi')\sin(\lambda' - \phi_{to})) \\
&= \arccos\left(\sqrt{\cos^2(\phi')\cos^2(\lambda' - \phi_{to}) + \sin^2(\phi')}\right) \\
&= \arccos\left(\sqrt{1 - \cos^2(\phi')\sin^2(\lambda' - \phi_{to})}\right).
\end{aligned}$$

However all these expressions except the first will be susceptible to loss of accuracy in numerical implementations.

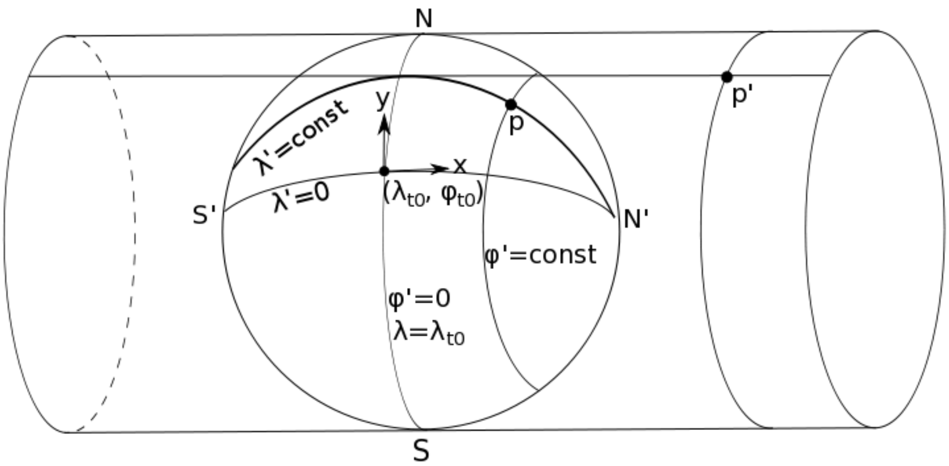


Figure 2: Illustration of a transverse mercator projection.

Note that, if $\lambda = \lambda_{to} \pm \pi/2$ and $\phi = 0$ (i.e. $\phi' = \pm\pi/2$), then the calculation of λ' gives $\arctan_2(0, 0)$ and is ill-defined. Also the point is projected to infinity and so it is impossible to calculate the corresponding x and y. In addition, if $\phi = \pm\pi/2$ (i.e. $\lambda' = \phi_{to} \pm \pi/2$ and $\phi' = 0$), then the calculation of λ gives $\arctan_2(0, 0)$ and is ill-defined. This requires some care in the numerical implementation.

In the transverse Mercator projection, infinitesimal distances are expanded by a factor of $1/\cos(\phi')$ relative to those on the sphere.

3.4 Transformations between a latitude-longitude coordinate system and a conformal Lambert projection based coordinate system

To convert a point (λ, ϕ) in a standard latitude-longitude coordinate system to the point (x, y) in a conformal Lambert coordinate system, we use the equations

$$x = \rho \sin[n(\lambda - \lambda_0)]$$

and

$$y = -\rho \cos[n(\lambda - \lambda_0)]$$

where

$$\rho = F \cot^n \left(\frac{\pi}{4} + \frac{\phi}{2}\right)$$

$$F = \frac{\cos\phi_1 \tan^n \left(\frac{\pi}{4} + \frac{\phi_1}{2}\right)}{n}$$

$$n = \frac{\ln(\cos\phi_1 \sec\phi_2)}{\ln\left[\tan\left(\frac{\pi}{4} + \frac{\phi_2}{2}\right)\cot\left(\frac{\pi}{4} + \frac{\phi_1}{2}\right)\right]}.$$

Here, the standard parallels of the conformal Lambert system are ϕ_1 and ϕ_2, the reference longitude is λ_0 and it is assumed that both coordinate systems have the same pole and origin.

In the case with one standard parallel ($\phi_1 = \phi_2$), we would instead use

$$n = \sin\phi_1$$

to calculate n, but NAME only supports conformal Lambert projections with two distinct standard parallels.

The value of n is related to the angle at the top of the unravelled cone, represented by α in figure 3. For example, in the case of a single standard parallel, $\alpha = 2\pi n$.

To convert from a conformal Lambert system to a latitude-longitude system, we use the equations

$$\phi = 2\tan^{-1}\left[\left(\frac{F}{\rho}\right)^{\frac{1}{n}}\right] - \frac{\pi}{2}$$

and

$$\lambda = \lambda_0 + \frac{\theta}{n}$$

where

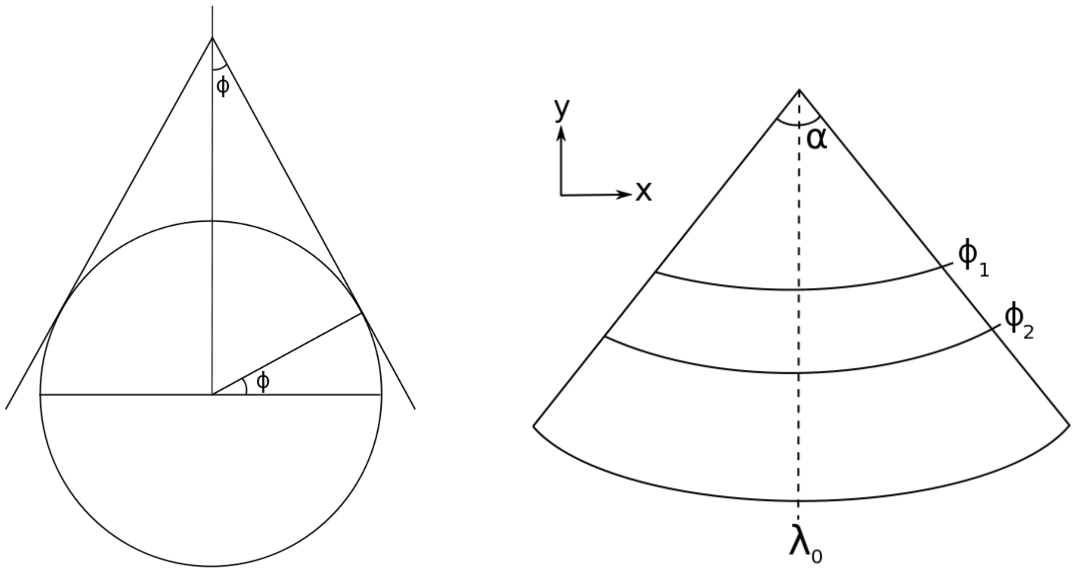
$$\rho = \mathrm{sgn}(n)\sqrt{x^2 + y^2}$$

and

$$\theta = \tan^{-1}\left(-\frac{x}{y}\right).$$

In the conversion calculations described above, the reference latitude is not used. In many sources of conversion formulae, it is assumed that the origin of the new coordinate system is at the intersection of the reference latitude and longitude. Within NAME, the position of the origin is calculated separately (in the same way that other points are converted) and then the conformal Lambert points entering or leaving the above formulae are adjusted according to these origin values.

When converting from a conformal Lambert point in metres, the input values must first be divided by the radius of the earth to bring it in to the correct units for the conversion. Conversely, when converting to a point in conformal Lambert, the values must be multiplied by the radius of the earth to give an result in metres.



(a) A cone placed over the earth with a single standard parallel ϕ.

(b) An unravelled cone with two standard parallels ϕ_1 and ϕ_2 and a reference longitude λ_0.

Figure 3: Illustration of conformal Lambert projections.

4 Differentials of transformations between different horizontal coordinate systems

Here we derive the results needed to transform vectors and covectors between the coordinate systems. Because all our horizontal coordinate systems are orthogonal we do not need the general techniques involving contravariant and covariant components (see e.g. Frankel (2004)) which are needed for arbitrary transformations. Instead we express the components of vectors and covectors in terms of their 'physical magnitudes' (e.g. metres per second instead of degrees latitude or longitude per second and Pascals per metre instead of Pascals per degree). As a result, all we need to know in order to transform between two coordinate systems is the local rotation angle between the coordinate lines.

However for some purposes it is useful to relate this to the more general approach. In the latter, the components are given, for vectors, by the coefficients multiplying the tangent vectors $\boldsymbol{\partial}_x$ and $\boldsymbol{\partial}_y$ to the s-parametrized curves $(x+s, y)$ and $(x, y+s)$ [i.e. by $(\dot{x}, \dot{y})$ for a tangent vector to the trajectory $(x(s), y(s))$] and, for covectors, by the coefficients multiplying the covectors dx and dy [i.e. by $(\partial f/\partial x, \partial f/\partial y)$ for the covector df]. The two approaches can be easily related by using the distances h_x and h_y which are travelled per unit change in coordinate along the curves $(x+s, y)$ and $(x, y+s)$. In addition we note that we can rescale an orthogonal coordinate system locally using h_x and h_y to make $\boldsymbol{\partial}_x$ and $\boldsymbol{\partial}_y$ orthonormal at any given point. When this is done the values of the contravariant and covariant components are equal to the physical magnitudes.

As in the previous section, it is straightforward to account for the scale factor, the origin offsets, the θ origin and the units. Hence we do not consider such matters here. For all the coordinate systems we assume that the scale factor is unity, that the origin offsets and the θ origin are zero, and that the units are radians and/or metres.

First we consider the values of h_x and h_y for the various coordinate systems. Following the assumption that the region of interest is thin relative to R_E, we neglect any variation of these quantities with height.

(i) Latitude-longitude systems:

$$h_\lambda = R_E \cos(\phi), \quad h_\phi = R_E.$$

(ii) Cartesian systems in stereographic projection:

$$h_x = h_y = \frac{2}{1+\sin(\phi)}$$

where ϕ is the latitude in the latitude-longitude system with the same values for λ_p, ϕ_p and e_3.

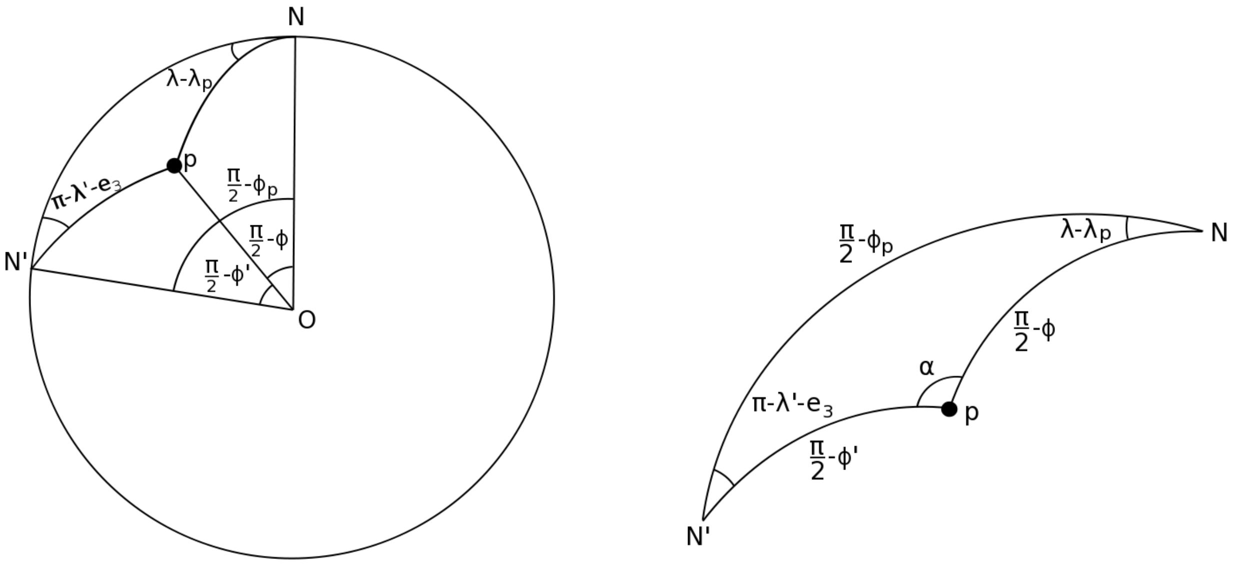


Figure 4: Illustration of the calculation of α for transformations between a latitude-longitude coordinate system and the standard latitude-longitude coordinate system.

(iii) Polar systems in stereographic projection:

$$h_r = \frac{2}{1+\sin(\phi)}, \quad h_\theta = \frac{2r}{1+\sin(\phi)}$$

where ϕ is the latitude in the latitude-longitude system with the same values for λ_p, ϕ_p and e_3.

(iv) Cartesian systems in transverse Mercator projection:

$$h_x = h_y = \frac{1}{\cos(\phi')}$$

where ϕ' is the latitude in the latitude-longitude system with $\lambda_p = \lambda_{to} + \pi/2$, $\phi_p = 0$ and $e_3 = -\phi_{to}$.

(v) Polar systems in transverse Mercator projection:

$$h_r = \frac{1}{\cos(\phi')}, \quad h_\theta = \frac{r}{\cos(\phi')}$$

where ϕ' is the latitude in the latitude-longitude system with $\lambda_p = \lambda_{to} + \pi/2$, $\phi_p = 0$ and $e_3 = -\phi_{to}$.

We now consider the local rotation angle α between a number of pairs of coordinate systems. The general case can be derived by combining these results.

(i) The angle between a latitude-longitude system and the standard latitude-longitude system. This is derived by considering the spherical triangles in figure 4 and using standard results for spherical triangles (see e.g. Abramowitz and Stegun (1976, §4.3.149); Weisstein (2003, page 2795-2798)). If we define α as positive when the latitude-longitude system is (locally) rotated anticlockwise relative to the standard latitude-longitude system, then we have

$$\sin(\alpha) = \sin(\pi/2 - \phi_p)\frac{\sin(\lambda - \lambda_p)}{\sin(\pi/2 - \phi')} = \sin(\pi/2 - \phi_p)\frac{\sin(\pi - \lambda' - e_3)}{\sin(\pi/2 - \phi)}$$

and

$$\cos(\alpha) = \sin(\lambda - \lambda_p)\sin(\pi - \lambda' - e_3)\cos(\pi/2 - \phi_p) - \cos(\lambda - \lambda_p)\cos(\pi - \lambda' - e_3).$$

α can then be calculated as $\arctan_2(\sin(\alpha), \cos(\alpha))$.

(ii) The angle between a stereographic projection based Cartesian system and a latitude-longitude system with the same values of λ_p, ϕ_p and e_3. If we define α as positive when the stereographic projection based Cartesian system is (locally) rotated anticlockwise relative to the latitude-longitude system, then we have

$$\alpha = -\lambda.$$

(iii) The angle between a transverse mercator projection based Cartesian system and a latitude-longitude system with $\lambda_p = \lambda_{to} + \pi/2$, $\phi_p = 0$ and $e_3 = -\phi_{to}$. If we define α as positive when the transverse mercator projection based Cartesian system is (locally) rotated anticlockwise relative to the latitude-longitude system, then we have

$$\alpha = \pi/2.$$

(iv) The angle between a polar system and a Cartesian system in the same projection. If we define α as positive when the polar system is (locally) rotated anticlockwise relative to the Cartesian system, then we have

$$\alpha = \theta.$$

5 Some particular horizontal coordinate systems

5.1 Coordinate systems used for the EMEP grids

The EMEP (European Monitoring and Evaluation Programme or Co-operative Programme for Monitoring and Evaluation of the Long-range Transmission of Air pollutants in Europe) grids are based on a stereographic projection with tangent point at the true North pole and y-axis pointing along the meridian at 32 deg W towards the north pole (i.e. $\lambda_p = 0$, $\phi_p = \pi/2$, $e_3 = -32°$). There are two EMEP grids (50 km and 150 km resolution) that are defined in terms of Cartesian coordinate systems in this tangent plane. In each case the grid squares are centred on locations with integer coordinate values (starting at 1) and the grid square boundaries are associated with half-integer coordinate values.

The 50×50 km^2 EMEP grid has a true grid length of 50 km at 60 deg N. This equates to a grid length of

$$50 \text{ km} \times \frac{2}{1 + \sin(60°)} = 53,589.84 \text{ m}$$

on the tangent plane, i.e. $x_u = y_u = 53,589.84$ m. The origin offset is $(x_o, y_o) = (-8x_u, -110y_u)$. The domain includes 132×111 points (with x varying from 1 to 132, and y varying from 1 to 111).

The 150×150 km^2 EMEP grid has a true grid length of 150 km at 60 deg N. This equates to a grid length of

$$150 \text{ km} \times \frac{2}{1 + \sin(60°)} = 160,769.52 \text{ m}$$

on the tangent plane, i.e. $x_u = y_u = 160,769.52$ m. The origin offset is $(x_o, y_o) = (-3x_u, -37y_u)$. The domain includes 44×37 points (with x varying from 1 to 44, and y varying from 1 to 37).

The 50 km and 150 km coordinates are related by $x_{50} = 3x_{150} - 1$ and $y_{50} = 3y_{150} - 1$ and the two sets of grid squares cover the same area.

The EMEP coordinate systems are defined using an assumed Earth radius of 6,370,000 m. As a result there are small errors in the above.[2]

Further information, including graphical plots of the two EMEP grids, is available on the EMEP website at www.emep.int.

5.2 UK National Grid

This is a Cartesian coordinate system using a transverse Mercator projection with true origin $(\lambda_{to}, \phi_{to}) = (2^o\text{W}, 49^o\text{N})$, scale factor $s = 0.9996012717$ and offset origin $(x_o, y_o) = (-400\ 000, +100\ 000)$. The units (x_u, y_u) equal $(1, 1)$ for the basic UK National Grid although sometimes coordinate values are divided by a power of 10 (i.e. x_u and y_u are multiplied by a power of 10) when lower precision is appropriate.

[2] This could be corrected by introducing a scale factor s as we have done for transverse Mercator projections.

As noted in §1 we assume that the Earth is spherical with a radius of 6,371,229 m. This leads to some errors relative to the true UK National Grid which assumes an ellipsoidal Earth (following the Airy 1830 ellipsoid). Some investigative calculations were conducted to compare true UK National Grid coordinates with those obtained for a spherical Earth with $R_E =$ 6,371,229 m. Computations were performed for various sample locations (specified by latitude and longitude), making use of the Ordnance Survey on-line calculator at www.gps.gov.uk. The errors were found to be at most a kilometre or so in the usual domain of the UK National Grid.

Sometimes the coordinates are reduced modulo 100,000 within 100 km squares, with the 100 km squares referred to by two-letter codes.

For further details see Ordnance Survey (2006) as well as the Ordnance Survey website at www.ordnancesurvey.gov.uk and www.gps.gov.uk.

5.3 Irish Grid

This is a Cartesian coordinate system using a transverse Mercator projection with true origin $(\lambda_{to}, \phi_{to}) = (8^o\text{W}, 53.5^o\text{N})$, scale factor $s = 1.000035$ and offset origin $(x_o, y_o) = (-200\ 000, -250\ 000)$. The units (x_u, y_u) equal $(1, 1)$ for the basic Irish Grid although sometimes coordinate values are divided by a power of 10 (i.e. x_u and y_u are multiplied by a power of 10) when lower precision is appropriate.

As noted in §1 we assume that the Earth is spherical with a radius of 6,371,229 m. This leads to some errors relative to the true Irish Grid which assumes an ellipsoidal Earth (following the Airy 1830 modified ellipsoid).

Sometimes the coordinates are reduced modulo 100,000 within 100 km squares, with the 100 km squares referred to by single-letter codes.

For further details see Ordnance Survey of Ireland and Ordnance Survey of Northern Ireland (2000) and Ordnance Survey (2006) as well as the Ordnance Survey of Ireland web site at www.osi.ie, the Ordnance Survey of Northern Ireland website at www.osni.gov.uk and the Ordnance Survey website at www.ordnancesurvey.gov.uk and www.gps.gov.uk.

6 Vertical coordinate systems

For convenience vertical coordinate systems are defined below ground (or sea) as well as above ground. Note that, in coordinate system definitions, pressure is always interpreted as dry hydrostatic pressure, i.e. we assume that the pressure is determined by the surface pressure and by $\partial p/\partial z|_{x,y} = -gp/RT$ where p is pressure, z is height, g is the acceleration due to gravity, R is the gas constant for dry air, and T is temperature. This ensures that pressure is monotonic with height and avoids complications due to moisture effects.

6.1 Height-based coordinate systems

6.1.1 Height above ground coordinate systems

The coordinate z_{agl} is defined to be height above ground level (in metres). Note that where, as here, we define a symbol to be a dimensionless quantity by specifying the units to be used, we may also wish to use the same symbol for the underlying dimensional quantity. Provided the meaning is clear from the context, we will do this without further comment.

6.1.2 Height above sea coordinate systems

The coordinate z_{asl} is defined to be height above mean sea level (in metres).

6.1.3 Analytic height-based eta coordinate systems[3]

The coordinate η is defined so that z_{agl} and z_{asl} are linear functions of η for heights below ground and heights above some 'interface height', and are quadratic functions of η in between, with continuous derivatives at the

[3]Support for non-analytic systems would also be useful in the future.

ground and at the interface height. η is related to z_{agl} and z_{asl} by

$$z_{agl} = \begin{cases} \eta(z_t - 2z_g/\eta_i) & \text{if } \eta \leq 0 \\ \eta z_t + (1-\eta/\eta_i)^2 z_g - z_g & \text{if } 0 < \eta < \eta_i \\ \eta z_t - z_g & \text{if } \eta \geq \eta_i \end{cases}$$

and

$$z_{asl} = \begin{cases} \eta(z_t - 2z_g/\eta_i) + z_g & \text{if } \eta \leq 0 \\ \eta z_t + (1-\eta/\eta_i)^2 z_g & \text{if } 0 < \eta < \eta_i \\ \eta z_t & \text{if } \eta \geq \eta_i \end{cases}$$

where z_t is the 'model top' (the height where $\eta = 1$), z_i is the interface height and z_g is the ground height (all heights in metres above mean sea level) and $\eta_i \equiv z_i/z_t$ is the value of eta at the interface. This can be inverted to give

$$\eta = \begin{cases} \dfrac{z_{agl}}{z_t - 2z_g/\eta_i} & \text{if } z_{agl} \leq 0 \\ -\dfrac{z_c}{z_g} + \left(\dfrac{z_c^2}{z_g^2} - 4\left(1 - \dfrac{z_{asl}}{z_g}\right)\right)^{1/2} & \text{if } z_{agl} > 0 \text{ and } z_{asl} < z_i \\ \dfrac{z_{asl}}{z_t} & \text{if } z_{asl} \geq z_i \end{cases}$$

where $z_c = z_i - 2z_g$. The middle range of this inversion formula is susceptable to precision error, overflow and division by zero, but can be re-expressed as

$$\begin{aligned} \eta/\eta_i &= -\frac{z_c}{z_g} + \left(\frac{z_c^2}{z_g^2} + 4\frac{z_{agl}}{z_g}\right)^{1/2} \\ &= \left[-\frac{z_c^2}{z_g^2} + \left(\frac{z_c^2}{z_g^2} + 4\frac{z_{agl}}{z_g}\right)\right] \Bigg/ \left[\frac{z_c}{z_g} + \left(\frac{z_c^2}{z_g^2} + 4\frac{z_{agl}}{z_g}\right)^{1/2}\right] \\ &= \frac{2z_{agl}}{(z_c^2 + 4z_g z_{agl})^{1/2} + z_c}. \end{aligned}$$

Note η equals 0 on the ground and increases with height.

For η to be well defined and to strictly increase with height we require $z_c > 0$ and hence that $z_g < z_i/2$.

6.2 Pressure-based coordinate systems

6.2.1 Pressure coordinate systems

The coordinate p is defined to be pressure (in Pascals).

6.2.2 Pressure-as-height coordinate systems

The coordinate z_{ICAO} is defined to be the height (in metres) which corresponds, in the ICAO standard atmosphere, to the pressure. The ICAO standard atmosphere is defined in terms of the following constants (Meteorological Office (1991, page 153))

- Temperature at 0km above mean sea level, $T_{msl} = 288.15$ K.
- Temperature at 11km above mean sea level $T_{11} = 216.65$ K.
- Temperature at 20km above mean sea level $T_{20} = 216.65$ K.
- Lapse rate from 0 to 11km above mean sea level $\gamma_{0-11} = 0.0065$ K m^{-1}.
- Lapse rate at more than 20km above mean sea level $\gamma_{20+} = -0.001$ K m^{-1}.
- Pressure at mean sea level $p_{msl} = 101,325.0$ Pa.

with constant lapse rate assumed between the fixed points.

Using the hydrostatic assumption and writing $z_{11} = 11000$ m, $p_{11} =$ pressure at 11km above mean sea level etc, we have

$$p_{11} = p_{msl}\left(1 - \frac{\gamma_{0-11}}{T_{msl}} z_{11}\right)^{g/R\gamma_{0-11}},$$

$$p_{20} = p_{11} \exp\left(-\frac{g}{RT_{11}}(z_{20} - z_{11})\right),$$

$$p = \begin{cases} p_{msl}\left(1 - \frac{\gamma_{0-11}}{T_{msl}} z_{ICAO}\right)^{g/R\gamma_{0-11}} & \text{for} \quad z_{ICAO} \le z_{11} \\ p_{11} \exp\left(-\frac{g}{RT_{11}}(z_{ICAO} - z_{11})\right) & \text{for} \quad z_{11} \le z_{ICAO} \le z_{20} \\ p_{20}\left(1 - \frac{\gamma_{20+}}{T_{20}}(z_{ICAO} - z_{20})\right)^{g/R\gamma_{20+}} & \text{for} \quad z_{ICAO} \ge z_{20}, \end{cases}$$

and

$$z_{ICAO} = \begin{cases} \frac{T_{msl}}{\gamma_{0-11}}\left(1 - \left(\frac{p}{p_{msl}}\right)^{R\gamma_{0-11}/g}\right) & \text{for} \quad p \ge p_{11} \\ z_{11} + \frac{RT_{11}}{g} \log\left(\frac{p_{11}}{p}\right) & \text{for} \quad p_{11} \ge p \ge p_{20} \\ z_{20} + \frac{T_{20}}{\gamma_{20+}}\left(1 - \left(\frac{p}{p_{20}}\right)^{R\gamma_{20+}/g}\right) & \text{for} \quad p \le p_{20}. \end{cases}$$

We will denote the temperature in the ICAO standard atmosphere at height z_{ICAO} by T_{ICAO}.

6.2.3 Pressure-based eta coordinate systems

The coordinate η is defined in terms of n 'specified levels', the level indices increasing with height. At the ith specified level, η and p are given by

$$\eta = A_i/p_{\eta\, ref} + B_i, \quad p = A_i + B_i p_s$$

where $p_{\eta\, ref}$ is a reference pressure and p_s is surface pressure (all pressures in Pascals). In between levels, η and p are linearly related. For convenience we will write $\eta_i = A_i/p_{\eta\, ref} + B_i$.

If $A = 0$, $\eta = p/p_s$ while if $B = 0$, $\eta = p/p_{\eta\, ref}$. The bottom level must have $A_1 = 0$ (and $B_1 \le 1$) so that it is 'terrain following' (and at or above ground). Also usually $B = 0$ near the top levels so that η is independent of p_s. Above the top level and below the bottom level η and p are assumed proportional. Equivalently we can imagine taking $A = B = 0$ as an extra level above the top level (if this level doesn't already exist) and any number of extra levels with $A = 0$, $B > B_1$ below the bottom level.

Note η equals 1 on the ground, decreases with height, and tends to zero at the top of the atmosphere.

η, and hence $A_i/p_{\eta\, ref} + B_i$, must strictly decrease with increasing level number. p must also strictly decrease with increasing level number, and this will be the case if B decreases (not necessarily strictly) with increasing level number and p_s exceeds $p_{s\, min} = \min_{\{i:B_{i+1}<B_i\}}(A_i - A_{i+1})/(B_{i+1} - B_i)$ (using the fact that η is strictly decreasing to deal with the case where $B_{i+1} = B_i$). We will require that B decreases (not necessarily strictly) with increasing level number but note that, if this were relaxed, the effect would be that there would be a $p_{s\, max}$ too. Note $p = \eta p_{\eta\, ref} + B(p_s - p_{\eta\, ref})$ (in fact this is true even away from the specified levels if B is interpolated vertically and linearly in either p or η), so B decreasing is necessary for the effect of p_s to decrease with height.

6.3 Units

For all the above coordinates we can use arbitrary units[4]. If we write $\hat{z}$ for the basic coordinate (i.e. z_{agl}, z_{asl}, analytic z-based η, p, z_{ICAO} or p-based η), then the new coordinate z is defined by $z = \hat{z}/u$ where u defines the unit to be used.

7 Transformations between different vertical coordinate systems

The transformations follow trivially from the above, except for cases involving a transformation between a height-based and a pressure-based coordinate system. Such transformations require knowledge of the state of the atmosphere and are not considered here.

[4] This is not yet true for the η coordinates but it will be extended to them in due course.

8 Differentials of transformations between different vertical coordinate systems

Here we derive the results needed to transform vectors and covectors between the coordinate systems. Although the horizontal coordinates are orthogonal, the full 3-d coordinate system is not generally orthogonal except when the vertical coordinate is z_{asl}. Hence, instead of considering rotation angles, we consider the partial derivatives $\partial z_1/\partial(x, y, z_2, t)$ where z_1 and z_2 are the two vertical coordinate systems of interest.

As in §3-4, it is straightforward to account for the use of arbitrary units. Hence we do not consider such matters here. For all coordinate systems we assume that u is unity. As in the previous section, transformations between a height-based and a pressure-based coordinate system require knowledge of the state of the atmosphere and are not considered here.

We start by noting some basic facts about differentials of transformations. For two coordinate systems (t, x, y, z) and (t', x', y', z') we denote the Jacobian matrix by

$$\frac{\partial(t', x', y', z')}{\partial(t, x, y, z)} \equiv \begin{pmatrix} \frac{\partial t'}{\partial t} & \frac{\partial t'}{\partial x} & \frac{\partial t'}{\partial y} & \frac{\partial t'}{\partial z} \\ \frac{\partial x'}{\partial t} & \frac{\partial x'}{\partial x} & \frac{\partial x'}{\partial y} & \frac{\partial x'}{\partial z} \\ \frac{\partial y'}{\partial t} & \frac{\partial y'}{\partial x} & \frac{\partial y'}{\partial y} & \frac{\partial y'}{\partial z} \\ \frac{\partial z'}{\partial t} & \frac{\partial z'}{\partial x} & \frac{\partial z'}{\partial y} & \frac{\partial z'}{\partial z} \end{pmatrix}.$$

If (t'', x'', y'', z'') is a third system we have

$$\frac{\partial(t'', x'', y'', z'')}{\partial(t', x', y', z')}\frac{\partial(t', x', y', z')}{\partial(t, x, y, z)} = \frac{\partial(t'', x'', y'', z'')}{\partial(t, x, y, z)} \tag{1}$$

from which it follows that

$$\frac{\partial(t, x, y, z)}{\partial(t', x', y', z')}\frac{\partial(t', x', y', z')}{\partial(t, x, y, z)} = I \tag{2}$$

where I is the unit matrix.

Here we are concerned with transformations where only the last coordinate changes. In this case we have

$$\frac{\partial(t, x, y, z')}{\partial(t, x, y, z)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\partial z'}{\partial t} & \frac{\partial z'}{\partial x} & \frac{\partial z'}{\partial y} & \frac{\partial z'}{\partial z} \end{pmatrix}$$

and we denote the bottom row of this matrix by $\partial z'/\partial(t, x, y, z)$. Then (1) yields

$$\frac{\partial z''}{\partial(t, x, y, z)} = \frac{\partial z''}{\partial(t, x, y, z')}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{\partial z'}{\partial t} & \frac{\partial z'}{\partial x} & \frac{\partial z'}{\partial y} & \frac{\partial z'}{\partial z} \end{pmatrix},$$

i.e.

$$\left.\frac{\partial z''}{\partial t}\right|_{x,y,z} = \left.\frac{\partial z''}{\partial t}\right|_{x,y,z'} + \left.\frac{\partial z''}{\partial z'}\right|_{t,x,y}\left.\frac{\partial z'}{\partial t}\right|_{x,y,z} \quad \text{and} \quad \left.\frac{\partial z''}{\partial z}\right|_{t,x,y} = \left.\frac{\partial z''}{\partial z'}\right|_{t,x,y}\left.\frac{\partial z'}{\partial z}\right|_{t,x,y}$$

(and of course similar results with t, x and y permuted). For the special case $z'' = z$ we have

$$0 = \left.\frac{\partial z}{\partial t}\right|_{x,y,z'} + \left.\frac{\partial z}{\partial z'}\right|_{t,x,y}\left.\frac{\partial z'}{\partial t}\right|_{x,y,z} \quad \text{and} \quad 1 = \left.\frac{\partial z}{\partial z'}\right|_{t,x,y}\left.\frac{\partial z'}{\partial z}\right|_{t,x,y}.$$

These results can of course also be obtained directly from (2) by using

$$\frac{\partial(t', x', y', z')}{\partial(t, x, y, z)} = \frac{\partial(t, x, y, z)}{\partial(t', x', y', z')}^{-1}$$

to obtain

$$\left.\frac{\partial z'}{\partial t}\right|_{x,y,z} = -\left.\frac{\partial z}{\partial t}\right|_{x,y,z'}\Big/\left.\frac{\partial z}{\partial z'}\right|_{t,x,y} \quad \text{and} \quad \left.\frac{\partial z'}{\partial z}\right|_{t,x,y} = 1\Big/\left.\frac{\partial z}{\partial z'}\right|_{t,x,y}.$$

Also the first result can be expressed equivalently as

$$\left.\frac{\partial z}{\partial z'}\right|_{t,x,y} \left.\frac{\partial z'}{\partial t}\right|_{x,y,z} \left.\frac{\partial t}{\partial z}\right|_{x,y,z'} = -1.$$

These results can be used to calculate various partial derivatives in terms of others whose values are already known.

8.1 Height-based coordinate systems

(i) $z_1 = z_{agl}$, $z_2 = z_{asl}$: For this case we have

$$\frac{\partial z_{agl}}{\partial t} = 0, \quad \frac{\partial z_{agl}}{\partial x} = -\frac{\partial z_g}{\partial x}, \quad \frac{\partial z_{agl}}{\partial y} = -\frac{\partial z_g}{\partial y}, \quad \frac{\partial z_{agl}}{\partial z_{asl}} = 1.$$

(ii) $z_1 = z_{agl}$, $z_2 =$ (analytic z-based) η: We consider three regimes. For $z_{agl} < 0$, $\eta < 0$ we have

$$\frac{\partial z_{agl}}{\partial t} = 0, \quad \frac{\partial z_{agl}}{\partial x} = -\frac{2\eta}{\eta_i}\frac{\partial z_g}{\partial x}, \quad \frac{\partial z_{agl}}{\partial y} = -\frac{2\eta}{\eta_i}\frac{\partial z_g}{\partial y}, \quad \frac{\partial z_{agl}}{\partial \eta} = z_t - \frac{2z_g}{\eta_i}.$$

For $z_g \le z_{agl} + z_g < z_i$, $0 \le \eta < \eta_i$ we have

$$\frac{\partial z_{agl}}{\partial t} = 0, \quad \frac{\partial z_{agl}}{\partial x} = -(1-r^2)\frac{\partial z_g}{\partial x}, \quad \frac{\partial z_{agl}}{\partial y} = -(1-r^2)\frac{\partial z_g}{\partial y}, \quad \frac{\partial z_{agl}}{\partial \eta} = z_t - \frac{2rz_g}{\eta_i}$$

where $r = 1 - \eta/\eta_i$. For $z_{agl} + z_g \ge z_i$, $\eta \ge \eta_i$ we have

$$\frac{\partial z_{agl}}{\partial t} = 0, \quad \frac{\partial z_{agl}}{\partial x} = -\frac{\partial z_g}{\partial x}, \quad \frac{\partial z_{agl}}{\partial y} = -\frac{\partial z_g}{\partial y}, \quad \frac{\partial z_{agl}}{\partial \eta} = z_t.$$

(iii) $z_1 = z_{asl}$, $z_2 = z_{agl}$: For this case we have

$$\frac{\partial z_{asl}}{\partial t} = 0, \quad \frac{\partial z_{asl}}{\partial x} = \frac{\partial z_g}{\partial x}, \quad \frac{\partial z_{asl}}{\partial y} = \frac{\partial z_g}{\partial y}, \quad \frac{\partial z_{asl}}{\partial z_{agl}} = 1.$$

(iv) $z_1 = z_{asl}$, $z_2 =$ (analytic z-based) η: We consider three regimes. For $z_{asl} - z_g < 0$, $\eta < 0$ we have

$$\frac{\partial z_{asl}}{\partial t} = 0, \quad \frac{\partial z_{asl}}{\partial x} = \left(1 - \frac{2\eta}{\eta_i}\right)\frac{\partial z_g}{\partial x}, \quad \frac{\partial z_{asl}}{\partial y} = \left(1 - \frac{2\eta}{\eta_i}\right)\frac{\partial z_g}{\partial y}, \quad \frac{\partial z_{asl}}{\partial \eta} = z_t - \frac{2z_g}{\eta_i}.$$

For $z_g \le z_{asl} < z_i$, $0 \le \eta < \eta_i$ we have

$$\frac{\partial z_{asl}}{\partial t} = 0, \quad \frac{\partial z_{asl}}{\partial x} = r^2\frac{\partial z_g}{\partial x}, \quad \frac{\partial z_{asl}}{\partial y} = r^2\frac{\partial z_g}{\partial y}, \quad \frac{\partial z_{asl}}{\partial \eta} = z_t - \frac{2rz_g}{\eta_i}$$

where $r = 1 - \eta/\eta_i$. For $z_{asl} \ge z_i$, $\eta \ge \eta_i$ we have

$$\frac{\partial z_{asl}}{\partial t} = 0, \quad \frac{\partial z_{asl}}{\partial x} = 0, \quad \frac{\partial z_{asl}}{\partial y} = 0, \quad \frac{\partial z_{asl}}{\partial \eta} = z_t.$$

(v) $z_1 =$ (analytic z-based) η, $z_2 = z_{agl}$: Here we calculate $\partial z_{agl}/\partial(t, x, y, \eta)$ and invert the matrix

$$\frac{\partial(t, x, y, z_{agl})}{\partial(t, x, y, \eta)}$$

to get

$$\frac{\partial \eta}{\partial t} = -\frac{\partial z_{agl}}{\partial t} \Bigg/ \frac{\partial z_{agl}}{\partial \eta} \quad \text{and} \quad \frac{\partial \eta}{\partial z_{agl}} = 1 \Bigg/ \frac{\partial z_{agl}}{\partial \eta}$$

where all partial derivatives on the left hand sides use the (t, x, y, z_{agl})-coordinates and all partial derivatives on the right hand sides use the (t, x, y, η)-coordinates, and the first equation remains true with t, x and y permuted.

(vi) $z_1 =$ (analytic z-based) η, $z_2 = z_{asl}$: Here we calculate $\partial z_{asl}/\partial(t,x,y,\eta)$ and invert the matrix

$$\frac{\partial(t,x,y,z_{asl})}{\partial(t,x,y,\eta)}$$

to get

$$\frac{\partial \eta}{\partial t} = -\frac{\partial z_{asl}}{\partial t}\bigg/\frac{\partial z_{asl}}{\partial \eta} \quad \text{and} \quad \frac{\partial \eta}{\partial z_{asl}} = 1\bigg/\frac{\partial z_{asl}}{\partial \eta}$$

where all partial derivatives on the left hand sides use the (t,x,y,z_{asl})-coordinates and all partial derivatives on the right hand sides use the (t,x,y,η)-coordinates, and the first equation remains true with t, x and y permuted.

(vii) $z_1 =$ (analytic z-based) $\eta \equiv \eta_1$, $z_2 =$ (analytic z-based) $\eta \equiv \eta_2$: Here we calculate $\partial z_{asl}/\partial(t,x,y,\eta_1)$ and $\partial z_{asl}/\partial(t,x,y,\eta_2)$ and use

$$\frac{\partial(t,x,y,\eta_1)}{\partial(t,x,y,\eta_2)} = \frac{\partial(t,x,y,z_{asl})}{\partial(t,x,y,\eta_1)}^{-1}\frac{\partial(t,x,y,z_{asl})}{\partial(t,x,y,\eta_2)}$$

to get

$$\left.\frac{\partial \eta_1}{\partial t}\right|_{x,y,\eta_2} = -\left.\frac{\partial z_{asl}}{\partial t}\right|_{x,y,\eta_1}\bigg/\left.\frac{\partial z_{asl}}{\partial \eta_1}\right|_{t,x,y} + \left.\frac{\partial z_{asl}}{\partial t}\right|_{x,y,\eta_2}\bigg/\left.\frac{\partial z_{asl}}{\partial \eta_1}\right|_{t,x,y}$$

and

$$\left.\frac{\partial \eta_1}{\partial \eta_2}\right|_{t,x,y} = \left.\frac{\partial z_{asl}}{\partial \eta_2}\right|_{t,x,y}\bigg/\left.\frac{\partial z_{asl}}{\partial \eta_1}\right|_{t,x,y}$$

where the first equation remains true with t, x and y permuted.

8.2 Pressure-based coordinate systems

(i) $z_1 = p$, $z_2 = z_{ICAO}$: Here $\partial p/\partial z_{ICAO}$ is the only non-zero partial derivative in $\partial p/\partial(t,x,y,z_{ICAO})$. It is given by

$$\frac{\partial p}{\partial z_{ICAO}} = \frac{dp}{dz_{ICAO}} = -\frac{gp}{RT_{ICAO}}$$

where we write dp/dz_{ICAO} to emphasise that p can be considered to be a function of z_{ICAO} alone.

(ii) $z_1 = p$, $z_2 =$ (p-based) η: For this case we have

$$\frac{\partial p}{\partial t} = B\frac{\partial p_s}{\partial t}, \quad \frac{\partial p}{\partial x} = B\frac{\partial p_s}{\partial x}, \quad \frac{\partial p}{\partial y} = B\frac{\partial p_s}{\partial y}, \quad \frac{\partial p}{\partial \eta} = \frac{(A_{i+1}-A_i)+(B_{i+1}-B_i)p_s}{\eta_{i+1}-\eta_i}$$

where the point lies between levels i and $i+1$ and B is vertically interpolated linearly in p or η.

(iii) $z_1 = z_{ICAO}$, $z_2 = p$: Here $\partial z_{ICAO}/\partial p$ is the only non-zero partial derivative in $\partial z_{ICAO}/\partial(t,x,y,p)$. It is given by

$$\frac{\partial z_{ICAO}}{\partial p} = \frac{dz_{ICAO}}{dp} = -\frac{RT_{ICAO}}{gp}.$$

(iv) $z_1 = z_{ICAO}$, $z_2 =$ (p-based) η: Here we calculate $\partial z_{ICAO}/\partial(t,x,y,p)$ and $\partial p/\partial(t,x,y,\eta)$ and use

$$\frac{\partial(t,x,y,z_{ICAO})}{\partial(t,x,y,\eta)} = \frac{\partial(t,x,y,z_{ICAO})}{\partial(t,x,y,p)}\frac{\partial(t,x,y,p)}{\partial(t,x,y,\eta)}$$

to get

$$\frac{\partial z_{ICAO}}{\partial t} = \frac{dz_{ICAO}}{dp}\frac{\partial p}{\partial t} \quad \text{and} \quad \frac{\partial z_{ICAO}}{\partial \eta} = \frac{dz_{ICAO}}{dp}\frac{\partial p}{\partial \eta}$$

where all partial derivatives use the (t,x,y,η)-coordinates and the first equation remains true with t, x and y permuted.

(v) $z_1 = $ (p-based) η, $z_2 = p$: Here we calculate $\partial p/\partial(t,x,y,\eta)$ and invert the matrix

$$\frac{\partial(t,x,y,p)}{\partial(t,x,y,\eta)}$$

to get

$$\frac{\partial \eta}{\partial t} = -\frac{\partial p}{\partial t} \bigg/ \frac{\partial p}{\partial \eta} \quad \text{and} \quad \frac{\partial \eta}{\partial p} = 1 \bigg/ \frac{\partial p}{\partial \eta}$$

where all partial derivatives on the left hand sides use the (t,x,y,p)-coordinates and all partial derivatives on the right hand sides use the (t,x,y,η)-coordinates, and the first equation remains true with t, x and y permuted.

(vi) $z_1 = $ (p-based) η, $z_2 = z_{ICAO}$: Here we calculate $\partial p/\partial(t,x,y,\eta)$ and $\partial p/\partial(t,x,y,z_{ICAO})$ and use

$$\frac{\partial(t,x,y,\eta)}{\partial(t,x,y,z_{ICAO})} = \frac{\partial(t,x,y,p)}{\partial(t,x,y,\eta)}^{-1} \frac{\partial(t,x,y,p)}{\partial(t,x,y,z_{ICAO})}$$

to get

$$\frac{\partial \eta}{\partial t} = -\frac{\partial p}{\partial t} \bigg/ \frac{\partial p}{\partial \eta} \quad \text{and} \quad \frac{\partial \eta}{\partial z_{ICAO}} = \frac{dp}{dz_{ICAO}} \bigg/ \frac{\partial p}{\partial \eta}$$

where all partial derivatives on the left hand sides use the (t,x,y,z_{ICAO})-coordinates and all partial derivatives on the right hand sides use the (t,x,y,η)-coordinates, and the first equation remains true with t, x and y permuted.

(vii) $z_1 = $ (p-based) $\eta \equiv \eta_1$, $z_2 = $ (p-based) $\eta \equiv \eta_2$: Here we calculate $\partial p/\partial(t,x,y,\eta_1)$ and $\partial p/\partial(t,x,y,\eta_2)$ and use

$$\frac{\partial(t,x,y,\eta_1)}{\partial(t,x,y,\eta_2)} = \frac{\partial(t,x,y,p)}{\partial(t,x,y,\eta_1)}^{-1} \frac{\partial(t,x,y,p)}{\partial(t,x,y,\eta_2)}$$

to get

$$\left.\frac{\partial \eta_1}{\partial t}\right|_{x,y,\eta_2} = -\left.\frac{\partial p}{\partial t}\right|_{x,y,\eta_1} \bigg/ \left.\frac{\partial p}{\partial \eta_1}\right|_{t,x,y} + \left.\frac{\partial p}{\partial t}\right|_{x,y,\eta_2} \bigg/ \left.\frac{\partial p}{\partial \eta_1}\right|_{t,x,y}$$

and

$$\left.\frac{\partial \eta_1}{\partial \eta_2}\right|_{t,x,y} = \left.\frac{\partial p}{\partial \eta_2}\right|_{t,x,y} \bigg/ \left.\frac{\partial p}{\partial \eta_1}\right|_{t,x,y}$$

where the first equation remains true with t, x and y permuted.

9 Some particular vertical coordinate systems

9.1 Flight level

This is defined to be z_{ICAO} but expressed in units of 100 feet (Meteorological Office (1991, page 116)), i.e. with $u = 100$ ft / m = 30.48 (Abramowitz and Stegun (1976, table 2.5, page 8); Wilson (2000)). The flight level is usually rounded to the nearest multiple of 10 (Meteorological Office (1991, page 116)), but this is not done in NAME III.

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